THE KURAMOTO MODEL FOR NOISY OSCILLATORS

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1 Introduction

The phenomenon of synchronization and its effects have been studied extensively as it has been observed for centuries in biological, chemical, physical and social systems. The various examples of oscillators found in nature include the rhythmical beating of our hearts, neuronal synchrony in the visual cortex, the synchronous flashing of fireflies, arrays of lasers, microwave oscillators and many more. One specific scenario in which synchronization effects have been studied to a great extent is that of phase or limit-cycle oscillators which consists of an ensemble of nonlinear oscillators moving in a globally attracting limit cycle of constant amplitude. Weak coupling of these oscillators gives rise to complex and important mathematical questions. Winfree was able to formulate the problem in terms of a huge population of interacting limit-cycle oscillators and undertake a phase reduction approach to the problem. He was also able to understand synchronization as a threshold process, meaning that with strong enough coupling a family of non-trivial synchronized equilibria exists. In this approach one can exploit the separation of timescales: a fast timescale for oscillators which relax to their limit cycles and a long timescale with weak coupling and slight frequency differences between the oscillators. It wasn't until 1975 that Kuramoto started working on collective synchronization. He used the perturbative method of averaging to show that for a large system of weakly coupled, nearly identical limit-cycle oscillators, the form of the phase equations is the following:

$$\dot{\theta} = \omega_i + \frac{1}{N}\sum_{i=1}^{N}\Gamma_{ij}\theta_j - \theta_i) \qquad i = 1...N \qquad (1) \tag{1}$$

where the i-th oscillator with natural frequency ω_i adjusts its phase velocity according to input from other oscillators through the pair-wise phase interaction functions Γ_{ij}. The natural frequencies ω_i are distributed according to a specified probability density g(ω) usually taken to be a symmetric, unimodal distribution such as a Lorentzian or a Gaussian with mean ω_0. The interaction functions Γ_{ij} can be thought as phase response of oscillator j to input from i.

This reduction to the phase model represents a tremendous simplification, in this formulation the topology (i.e ring, cubic lattice, random graph) and the form of the phase response curve remain unspecified which makes a general analysis of the model far too difficult. The classic Kuramoto model specifies global (all-to-all) coupling mediated by a purely sinusoidal interaction function:

$$\Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N}\sin(\theta_j - \theta_i) \qquad \dot{\theta} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \qquad i = 1...N \tag{2}$$

with K being the coupling strength and $\frac{1}{N}$ ensuring that the model is well behaved when $N \to \infty$.

The sinusoidal interaction function is a first-order approximation to the more general form of (1) but still permits a variety of highly non-trivial solutions. A notable feature of this choice is that the interaction function vanishes when the phases are identical (oscillators

are in phase or anti-phase). In the case of near antiphase, the phases are pushed apart, meaning that there exists a single attracting synchronous and a single unstable antiphase constellation for pairs of oscillators. This model is the canonical form for synchronization.
In the Kuramoto model, the impact of increasing K is specifically the increase the phase synchrony amongst the oscillators. For weak K the oscillators disperse, whereas for strong K they remain relatively synchronous. In the case of intermediate K, it can be observed that a large cluster of synchronous oscillators appear. As K increases, the interaction functions overcome the dispersion of natural frequencies ω_i resulting in a transition from incoherence, to partial and then full synchronization. In order to quantify the degree of synchrony, the formula below was employed in order to calculate the centroid vector of this phase distribution:

$$re^{i\psi} = \frac{1}{N}\sum_{i=1}^{N} e^{i\theta_j}$$

where ψ is the mean phase of the set of θ_j and the scalar r represents the phase divergence or uniformity. Phase coherence r is identified as the order parameter of the system: it vanishes when the individual oscillations add incoherently and no macroscopic rhythm is produced and approaches one when the oscillators move in a tight clump and their phases become aligned.

As Kuramoto observed, we can rewrite $\dot{\theta} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$ in terms of the order parameter by multiplying both sides by $e^{-i\theta_i}$

$$re^{i(\psi-\theta_i)} = \frac{1}{N}\sum_{i=1}^{N} e^{i(\theta_j-\theta_i)}$$

Equating imaginary parts yields:

$r(t)\sin(\psi(t) - \theta_i(t)) = \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)), \qquad i = 1, ..., N$

The system now becomes:

$$\dot{\theta}_i = \omega_i + \kappa r \sin(\psi - \theta_i) \qquad i = 1, ..., N$$

Based on this formulation, each oscillator is independent and uncoupled from all the others and depend of the mean field alone through which they still interact. For a greater phase coherence (larger r) there's an increase in the effective adjustment of each oscillator's phase toward the mean field which thus leads to further increases in phase coherence. Kuramoto exploited this representation to derive an analytic value for K_c.
We can assume g(ω) to be a Gaussian or some other density with infinite tails and observe how the coupling K varies. For all K less than a certain threshold K_c, the oscillators act as if they were uncoupled, but when K exceeds K_c, this incoherent state becomes unstable, r(t) grows exponentially and a small cluster of oscillators that are mutually synchronized appears, in this way generating a collective oscillation. The population of oscillators splits into two groups: the oscillators near the center of the frequency distribution lock together at the mean frequency while those in the tails run near their natural frequencies and drift

relative to the synchronized cluster. This mixed state is often called partially synchronized. With further increases in K, more and more oscillators are recruited into the synchronized cluster.

2 Stochasticity

All the dynamics discussed thus far incorporate stochasticity solely through the randomness of the oscillators' frequencies ω_i. Now we include stochastic forces by explicitly introducing white noise into the dynamics. In order to do so, we return to the finite N ensemble of phase oscillators and write the so-called Langevin equation of the stochastic Kuramoto model:

$$\dot{\theta} = \omega_i + \xi_j(t) + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \qquad i = 1, ..., N \tag{3}$$

where the $\xi_j(t)$ are spatially independent and temporally uncorrelated random fluctuations with vanishing means and variance σ^2. Once again, by recasting the model by using the mean field formulation, in the presence of stochastic forces the model now becomes:

$$\dot{\theta}_i = \omega_i + \xi_j(t) + \kappa r \sin(\psi - \theta_i) \qquad i = 1, ..., N \tag{4}$$

We now introduce the probability density function for the oscillators at time t with natural frequency $\omega, \rho(t, \theta, \omega)$. Let $\rho(\theta, t, \omega)d\theta$ denote the fraction of these oscillators that lie between θ and $\theta + d\theta$ at time t. Here, ρ is nonnegative, 2π-periodic in θ and normalized, $\int_0^{2\pi} \rho(t, \theta, \omega)d\theta = 1$

The parameter order is now introduced by:

$r(t)e^{i\psi(t)} = \int_R \int_0^{2\pi} e^{i\tilde{\theta}}\rho(t, \tilde{\theta}, \omega)g(\omega)d\tilde{\theta}d\omega$ and $r(t)(\sin\psi(t) - \theta) = \int_R \int_0^{2\pi} (\sin(\tilde{\theta} - \theta)\rho(t, \tilde{\theta}, \omega)d\tilde{\theta}d\omega$

The evolution of ρ is governed by the continuity equation:

$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \rho}(\rho U[\rho]$ which expresses conservation of oscillators of frequency ω. Here the velocity $U[\rho](t, \theta, \omega]$ is interpreted in an Eulerian sense as the instentaneous velocity of an oscillator at position θ, given that it has natural frequency ω.

$$\begin{aligned} U[\rho](t, \theta, \omega) &= \omega + \kappa r(t)\sin(\psi(t) - \theta) \\ &= \omega - \kappa \int_R (\sin * \rho)(t, \theta, \omega)g(\omega)d\omega \\ &= \omega - \kappa \int_R \int_0^{2\pi} \sin(\theta - \tilde{\theta})\rho(t, \tilde{\theta}, \omega)g(\omega)d\tilde{\theta}d\omega. \end{aligned}$$

The white nose variables $\xi_j(t)$ introduced above are independent processes that satisfy:

$$\langle \xi_j(t) \rangle = 0, \qquad < \xi_j(s)\xi_j(t) >= 2D\delta_{ij}\delta(s-t)$$

Here $D \geq 0$ is the noise strength and the angular brackets denote an average over realizations of the noise. As argued by Sakaguchi, since we have a system of Langevin equations with mean-field coupling, as $N \to \infty$ the density $\rho(\theta, t, \omega)$ should satisfy the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}(\rho U[\rho])$$

Thus, the Sakaguchi's Fokker Planck equation reduces to the continuum limit of the Kuramoto model when $D = 0$.

3 Two-mode PIF

However, while the Kuramoto model provides a simple way of modeling coupled oscillators, the simple sinusoidal interaction function that is employed in it may not be a typical functional form one could find in wide ranges of experimental situations . An example would be the Hodgkin-Huxley model of neurons representing real spiking neurons of the brain which are coupled electrically. In this case a simple sinusoidal function is not sufficient, but at least three Fourier modes are necessary. When considering the two-mode PIF of the form:

$$\Gamma(\theta) = \sin(\theta) + 2\epsilon \sin 2(\theta - \alpha), \tag{5}$$

the parameter ϵ modifies the contribution of the second order term and α de-phases the relative postion of the two modes.

The first thing to note is that the two modes have opposite sign and the Kuramoto model with a one-mode PIF features one stabe and one unstable fixed point. In the two-mode PIF the two modes intersect twice and therefore the PIF vanishes at four points along a full cycle: two stable (attracting) nodes separate two unstable saddles. These extra fixed points therefore facilitate the existence of distinct clusters of phase locked oscillators even if the coupling is otherwise global, a phenomenon that is not possible with the first Fourier mode alone. This dynamic instability in the PIF enables a rich variety of more complex dynamics, most notably the emergence of heteroclinic cycles which connect a a pair of two-cluster states. The two cluster states appear to be unstable, however they are sensitive to noise. The injection of a stochastic influence into the states stabilizes the frequency of the slow heteroclinic cycling which then scales with log of the variance noise. In these systems, two distinct time scales arise naturally: the fast dynamics of the oscillators and the relatively slow rotation through the heteroclinic cycle. The two cluster states appear unstable, they are sensitive to noise hence the slow periodic oscillation between them: a limit cycle featuring yet another timescale for the system.

We can define:

$$U[\rho](t,\theta) = -\kappa \int_0^{2\pi} \Gamma(\theta-\theta')\rho(t,\theta')d\theta' = -\kappa(\Gamma * \rho)(t,\theta).$$

We introduce now the interaction potential W by $\partial_\theta W = \rho$, and $V[\rho] = \kappa(W * \rho)$, we have $-\partial_\theta V = U$ and we can write the equation as follows:

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial \theta^2} + \frac{\partial}{\partial \theta}(\rho \frac{\partial}{\partial \theta} V[\rho]),$$

or

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \theta}\left(e^{-V[\rho]}\frac{\partial}{\partial \theta}(\rho e^{V[\rho]})\right)$$

from which we get the equation for the steady states as follows:

$$\rho = Z^{-1}e^{-W*\rho}, \quad Z = \int_0^{2\pi} e^{-W*\rho}d\theta,$$

When considering the two mode PIF of the form :

$$\Gamma(\theta) = \sin\theta + 2\epsilon\sin(\theta-\alpha)$$

it is clear that:

$$W(\theta) = -(\cos\theta + \epsilon\cos(\theta-\alpha)$$

therefore:

$$\begin{aligned} V[\rho](\theta) &= \kappa(W * \rho)(\theta) \\ &= -\kappa[\langle\cos\theta\rangle_\rho \cos\theta + \langle\sin\theta\rangle_\rho \sin\theta] \\ &= -\kappa\epsilon[\langle\cos 2\theta\rangle_\rho \cos 2(\theta-\alpha) + \langle\sin 2\theta\rangle_\rho \sin 2(\theta-\alpha)] \end{aligned} \tag{6}$$

Futhermore, we can introduce the following:

$$r\exp(\theta_0 i) = \kappa\langle\exp(\theta i)\rangle_\rho \tag{7}$$

$$s\exp(2\tau_0 i) = \kappa\epsilon\langle\exp(2\theta i)\rangle_\rho \tag{8}$$

It follows that:

$$r = \kappa\langle\cos(\theta-\theta_0)\rangle_\rho \tag{9}$$

$$s = \kappa\epsilon\langle\cos 2(\theta-\tau_0)\rangle_\rho \tag{10}$$

$$0 = \langle\sin 2(\theta-\theta_0)\rangle_\rho \tag{11}$$

$$0 = \langle\sin 2(\theta-\tau_0)\rangle_\rho \tag{12}$$

If we convert into polar coordinates we get:

$$r_1 = r\cos\theta_0, \qquad r_2 = r\sin\theta_0 s_1 = s\cos 2\tau_0, \qquad s_2 = s\sin\tau_0 \tag{13}$$

For

$$\phi = \theta - \theta_0, \beta = \theta_0 - \tau_0, \gamma = \alpha - \beta$$

we get the following:

$$\begin{aligned} V[\rho](\theta) &= -[r_1\cos\theta + r_2\sin\theta + s_1\cos(\theta-\alpha) + s_2\sin 2(\theta-\alpha) \\ &= -[r\cos(\theta-\theta_0) + s\cos 2(\theta-\tau_0-\alpha) \\ &= -[r\cos\phi + s\cos 2(\phi-\gamma)] \end{aligned} \tag{14}$$

Now, the steady state equation becomes:

$$\rho(\theta) = Z^{-1}\exp(r\cos\phi + s\cos 2(\phi-\gamma)), \qquad Z = \int_0^{2\pi} \exp(r\cos\phi + s\cos 2(\phi-\gamma))d\phi. \tag{15}$$

Next, we define the generalized von Mises distribution which provides a flexible model for circular data allowing for symmetry, asymmetry, unimodality and multimodality.

Definition 3.1. $f : [0,2\pi)^2 \times \mathbb{R}^2 \times [0,\pi] \to \mathbb{R}$

$$(\theta|\theta_0, r, s, \gamma) \to Z^{-1}\exp(r\cos(\theta-\theta_0) + s\cos 2(\theta-\theta_0-\gamma)),$$

$$Z = \int_0^{2\pi} (\exp(r\cos\phi + s\cos 2(\phi-\gamma))d\phi.$$

For r,s $\in \mathbb{R}, \gamma \in [0,\pi)$ and n=0,1,2... the generalized modified Bessel functions can be defined as follows:

$$C_n(r,s,\gamma) = \frac{1}{\pi}\int_0^{2\pi} \cos n\phi \mu(\phi,r,s,\gamma)d\phi \tag{16}$$

$$S_n(r,s,\gamma) = \frac{1}{\pi}\int_0^{2\pi} \sin n\phi \mu(\phi,r,s,\gamma)d\phi \tag{17}$$

where $\mu(\phi,r,s,\gamma) = \exp(r\cos\phi + s\cos 2(\phi-\gamma))$.

Remark 3.1. *If $s=0, \gamma=0$ or $\gamma=\pi/2$, $\mu(\phi,r,s,\gamma)$ is even in ϕ, so $S_n(r,0,\gamma)=0$ for $n \in \mathbb{N}$. Also, for $s=0, C_n(r,s,\gamma) = I_n(r)$ where*

$I_n(r) = \frac{1}{\pi}\int_0 2\pi\cos n\phi\, \exp(r\cos\phi)d\phi$, is the modified Bessel function of order n. Instead, for $\gamma=0$ or $\gamma=\pi/2$; $C_n(r,s,\gamma) = I_n(r,s)$, or $C_n(r,s,\gamma) = I_n(r,-s)$ where

$$I_n(r_1,r_2) = \frac{1}{\pi}\int_0^{2\pi} \cos n\phi \exp(r_1\cos\phi + r_2\cos 2\phi)d\phi$$

is the generalization of the modified Bessel function of two dimensions.

S_n and C_n are even in r for even n and odd in r for odd n.

Also, for odd n

$$S_n(0, s, \gamma) = C_n(0, s, \gamma) = 0$$

Proposition 3.1. *A probability distribution ρ satisfies the steady state equation (3) if and only if it is a GVMD:*

$$\rho(\theta) = f(\theta|\theta_0, r, s, \gamma)$$

where $\theta_0 \in [0, 2\pi)$ and s, r, and γ are determined by:

$$\kappa \frac{C_1(r, s, \gamma)}{C_0(r, s, \gamma)} = r \tag{18}$$

$$S_1(r, s, \kappa) = 0 \tag{19}$$

$$\kappa\epsilon \frac{C_2(r, s, \gamma)}{C_0(r, s, \gamma)} = s \cos 2(\gamma - \alpha) \tag{20}$$

$$\kappa\epsilon \frac{S_2(r, s, \gamma)}{C_0(r, s, \gamma)} = s \sin 2(\gamma - \alpha) \tag{21}$$

Theorem 3.1. *For $\epsilon > 0$ and $\alpha \notin 0, \pi/2$, the steady state equation allows only for incoherent solutions $\rho = 1/2\pi$.*
For $\epsilon = 0$ which is the Kuramoto case, equations (7) to (10) reduce to:

$$\frac{I_1(r)}{rI_0(r)} = \frac{1}{\kappa}$$

For $\alpha = 0$, equations (7) to (10) reduce to:

$$\frac{I_1(r, \pm s)}{I_0(r, \pm s)} = \frac{r}{\kappa}$$

$$\frac{I_2(r, \pm s)}{I_0(r, \pm s)} = \frac{\pm s}{\kappa\epsilon}$$

Proof. For $s = 0$, the equation (17) requires either $\epsilon = 0$ in the Kuramoto case, or $I_2(r) = 0$, which implies $r = 0$ and hence $\rho = \frac{1}{2\pi}$
For $r = 0$, solutions to (15) and (16) are trivial. Hoewever, if we compute the following: (18) $\times \sin 2\gamma -$ (17) $\times \cos 2\gamma$ we get:

$$\kappa\epsilon \int_0^{2\pi} \frac{\sin 2(\phi - \gamma)\mu(\phi, 0, s, \gamma)d\phi}{\int_0^{2\pi} \mu(\phi, 0, s, \gamma)d\phi} = -s \sin 2\alpha. \tag{22}$$

The LHS is zero, hence either $s = 0$ and $\rho = \frac{1}{2\pi}$, $\alpha = 0$ or $\alpha = \frac{\pi}{2}$.
For $\gamma = 0$ or $\gamma = \pi/2$, $\mu(\phi, r, s, \gamma)$ is even in ϕ, therefore $S_2(r, s, \gamma) = 0$. Also, from (18), $\sin 2\alpha = 0$, so either $\alpha = 0$, or $\alpha = \pi/2$.

□

Lemma 3.1. *$S_1(r,s,\gamma)$ =0 if and only if one of the following statements is true: γ = 0, γ = 0, r = 0, s = 0.*

Proof. From Remark 1, we know that for $\gamma = 0$, $\gamma = 0$, $r = 0$, $s = 0$, $S_1(r,s,\gamma)$ =0 . Therefore, the 'only if' part of the Lemma follows from the remark.
For the 'if' part, $S_1(r,0,\gamma) = S_1(0,s,\gamma) = 0$, r,$s \geq 0$. Additionally,

$$\partial_r S_1 = \int_0^{2\pi} \cos\phi \sin\phi \mu(\phi) d\phi, \tag{23}$$

$$\partial_2^2 S_1 = \int_0^{2\pi} \cos^2\phi \sin\phi \mu(\phi) d\phi,$$

Moreover,

$$\begin{aligned}\partial_s S_1 &= \int_0^{2\pi} \sin\phi \cos 2(\phi-\gamma)\mu(\phi)d\phi \\ &= \cos 2\gamma \int_0^{2\pi} \sin\phi(2\cos^2\phi - 1)\mu(\phi)d\phi + \sin 2\gamma \int_2^{2\pi} 2\sin^2\phi\cos\phi\mu(\phi)d\phi. \\ &= \cos 2\gamma[2\partial_r^2 S_1 - S_1] + \sin 2\gamma \int_0^{2\pi} 2\sin^2\phi\cos\phi\mu(\phi)d\phi.\end{aligned}$$

Futhermore, we introduce the following:

$$W = \exp(s\cos 2\gamma)S_1,$$

it follows that

$$\partial_s W = \nu \partial_r^2 W + g,$$

where $\nu(\gamma) = 2\cos 2\gamma$, also:

$$\begin{aligned}g(r,s,\gamma) &= 2\sin 2\gamma \exp(s\cos 2\gamma) \int_0^{2\pi} \sin^2\phi\cos\phi\mu(\phi,r,s,\gamma)d\phi \\ &= 2\sin 2\gamma \exp(s\cos 2\gamma)[C_1(r,s,\gamma) - C_3(r,s,\gamma)].\end{aligned}$$

We know that $C_1 - C_3$ is odd in r. Also,

$$\partial_r \int_0^{2\pi} \sin^2 \cos\phi\mu(\phi)d\phi = \int_0^{2\phi} \sin^2\phi\cos^2\phi\mu(\phi)d\phi > 0.$$

Therefore, $W(r,0,\gamma) = W(r,s,\gamma) = 0$, with a forcing g such that fpr $r,s > 0$ has the same sign as $\sin 2\gamma$ and it is zero is and only if $\gamma = 0$ or $\gamma = \pi/2$. From the maximum principle if follows that for $r,s > 0$, $W(r,s\gamma)$ and also $S_1(r,s,\gamma)$ have the same sign as $\sin 2\gamma$ and are zero if and only if $\gamma = 0$ or $\gamma = \pi/2$. □

We consider the cases when $\alpha \in 0, \pi/2$. In the Kuramoto case, $\epsilon = 0$, we are left with the following equation:

$$\mathcal{G}(r) = \frac{1}{\kappa} \tag{24}$$

where the $\mathcal{G}(r)$ is defined as follows:

$$\mathcal{G}(r) = \frac{d}{rdr} \log I_0(r) = \frac{I_1(r)}{rI_0(r)}. \tag{25}$$

Proposition 3.1. *(a) The function* $r \to \frac{I_0''}{I_0(r)}$ *is positive and strictly increasing on* $(0, \infty)$*;*

(b) $\mathcal{G}(0) = \frac{1}{2}$ *and* $\mathcal{G}$ *is even and strictly decreasing on* $(0, \infty)$*.*

(c) The equation has a unique solution $\mathcal{G}^{-1}(1/\kappa) > 0$ *for* $\kappa > 2$*.*

Proof. (a) Since $\left(\frac{I_0''}{I_0}\right)'(0) = 0$.

(b) To see that $\mathcal{G}$ is even, we integrate by parts and obtain the following:

$$\mathcal{G}(r) = 1 - \frac{I_0''}{I_0} = \frac{1}{2}\left(1 - \frac{I_2}{I_0}\right),$$

by applying part (a), we conclude that $\mathcal{G}$ is strictly decreasing on $(0, \infty)$.

□

Proposition 1. *For* $\kappa \leq (1+\epsilon)^{-1}$*, the trivial solution* $(r, s) = 0$ *is the unique solution in* $\mathbb{R}^2$*. It is also the unique solution in* $[0, \infty)^2$ *for* $\kappa \leq 2(1+\epsilon)^{-1}$*.*

Proof. If we integrate by parts we get the following:

$$r = \kappa \int_{-\pi}^{\pi} (r \sin^2 \phi + 2s \sin \phi \sin 2\phi)\mu(\phi)d\phi \tag{26}$$

$$2s = \kappa\epsilon \int_{\pi}^{-\pi} (r \sin \phi \sin 2\phi + 2s \sin^2(2\phi))\mu(\phi)d\phi \tag{27}$$

hence it follows that

$$\begin{aligned}
r^2 + 4s^2/\epsilon &= \kappa \int_{-\pi}^{\pi} (r \sin \phi + 2s \sin(2\phi))^2 \mu(\phi)d\phi \\
&\leq \kappa \int_{-\pi}^{\pi} (1+\epsilon)\left(r^2 \sin^2 \phi + \frac{4s^2}{\epsilon} \sin^2 2\phi\right)\mu(\phi)d\phi \\
&= \kappa(1+\epsilon)\left(r^2 \int_{-\pi}^{\pi} \sin^2 \phi\mu(\phi)d\phi + \frac{4s^2}{\epsilon} \int_{-pi}^{\pi} \sin^2 2\phi\mu(\phi)d\phi\right) \\
&= \frac{\kappa(1+\epsilon)}{2}\left(r^2\left(1 - \frac{I_2}{I_0}\right) + \frac{4s^2}{\epsilon}\left(1 - \frac{I_4}{I_0}\right)d\phi\right),
\end{aligned} \tag{28}$$

□

Proposition 1. *If $\epsilon \neq 1$, then the map $\mathcal{F}(\kappa, r, s) = \nabla_{(r,s)}\ \nu(\kappa, r, s)$ has two pitchfork bifurcation points, (2,0) and (2/ϵ,0). Then, γ_1 is a supercritical bifurcating branch for $\epsilon \in$ (0,1/2) $\cup$ $(1, \infty)$ and subcritical for $\epsilon \in$ (1/2,1). The other branch γ_2 is supercritical.*

Proof.

$$\nabla^2_{(r,s)}\nu(\kappa, 0) = \begin{pmatrix} 1 - \frac{\kappa}{2} & 0 \\ 0 & \frac{1}{\epsilon} - \frac{\kappa}{2} \end{pmatrix} \tag{29}$$

has eigenvectors e_1 = (1,0) and e_2 = (0,1) and is singular for $\kappa = \kappa_1 = 2$ and $\kappa = \kappa_2 = 2/\epsilon$. We can conclude the existence of the birfucation branches γ_i for $i = 1, 2$ with an parametrization $I_i \ni t \to (\kappa_i(t), r_i(t), s_i(t))$ where $I_i \ni 0$ is an open interval. The Taylor series representation are:

$$\kappa_i(t) = \kappa_i + \sum_{j=1}^{\infty} k_i^{(j)} t^j$$

$$(r_i(t), s_i(t)) = e_i t + \sum_{j=1}^{\infty} v_i^{(j)} t^j$$

where $k_i^{(j)} \in \mathbb{R}$ and $v_i^{(j)} \in \mathbb{R}^2$. Also, $k_i^{(1)} = 0$ and

$$k_1^{(2)} = \frac{1/2 - \epsilon}{1 - \epsilon}; k_2^{(2)} = \frac{1}{2\epsilon}.$$

$k_i^{(2)} > 0$ indicates a supercritical bifurcation and $k_i^{(2)} < 0$ indicates a subcritical bifurcation. $\square$

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