STOCHASTIC POPULATION DYNAMICS

HONORS THESIS
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WITH HONORS

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I certify that I have read this honors thesis and that, in my opinion, it is fully adequate in scope and quality as an honors thesis for the degree of Bachelor of Science in Mathematics.

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I certify that I have read this honors thesis and that, in my opinion, it is fully adequate in scope and quality as an honors thesis for the degree of Bachelor of Science in Mathematics.

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Abstract

Analyzing population dynamics is vital to the existence of a sustainable planet. There is an undeniable relationship between population dynamics and global environmental changes, and there exists a growing need for a more thorough understanding of these connections. Population growth, composition, and distribution have important interrelationships with land use, land cover, and global climate change.

Stochastic processes arise commonly in nature. Many stochastic processes behave, at least for long stretches of time, like random walks with small but frequent jumps. The arguments discussed suggest that such processes will look, at least approximately, and on the appropriate time scale, like Brownian motion. The significance of the Logistic model and potential stochastic effects on population dynamics are discussed and approached using simulations of random walks and Brownian motion. This project will delve into mathematically incorporating stochasticity into population dynamics.

Chapter 1

Population Growth Models

1.1 The Logistic Model

In many cases, the rate of change of a process is not constant, but depends on the current state of the system (e.g. the rate of population growth depending on the current population size). Equations that relate a function to one or more of its derivatives are called *differential equations*. However, many functions needed to describe a system are unknown, yet may be found through their rate of change, i.e. their derivative(s).

The *Logistic model* is a model of population growth in which the growth rate of a population is proportional to population size. Specifically, logistic growth occurs in situations where the rate of change of a population P is proportional to the product of the number present at any time $P(t)$ and the difference between the number present and a number K denoting the carrying capacity, or the maximum potential population density. Note: $K > 0$.

This rate of change may be expressed as the differential equation:

$$\frac{dP}{dt} = kP(1 - \frac{P}{K})$$

which is often known as the Verhulst model. Here, k is a positive parameter representing the base rate of population growth which decreases as the population approaches its stable maximum size. This nonlinear equation may be solved by separation of variables and then a method of partial fractions resulting in solutions:

$$P(t) = \frac{K}{1 + Ae^{-kt}}$$

where $A = \frac{K - P_0,}{P_0}$.

1.2 Carrying Capacity

Population growth can be simply modeled without limits using the exponential function. Exponential growth occurs when a population's growth rate increases over time proportional to the size of the population without bounds.
A general equation expressing exponential growth is modeled by the following equation:

$$\frac{dP}{dt} = kP$$

As we know, the solution to this differential equation is a function $P(t)$ that is proportional to the exponential function

$$P(t) = P_0 e^{kt}$$

where $P_0 = P(0)$.

This equation models population growth most accurately during early stages, while resources are seemingly unlimited. Consider a small bacteria sample in a large Petri dish. When population dynamics continuously play out in nature, however, growth eventually faces restrictions and constraints. Any resource vital to a species' survival may serve as a limiting factor, such as finite space or food supply. Competition for resources as well as inherent limiting factors of the environment ensure that no population can continue to grow indefinitely. The *carrying capacity* of a population is defined as the maximum number of organisms which a finite environment may support indefinitely. Contrary to the exponential growth model where the per capita growth rate k is constant even when populations become extremely large, the per capita growth rate k of the logistic model decreases as the population approaches its maximum size. This behavior is illustrated by the following graph with initial population $P_0 = 20$, carrying capacity $K = 200$, and growth rate $k = 0.03$:

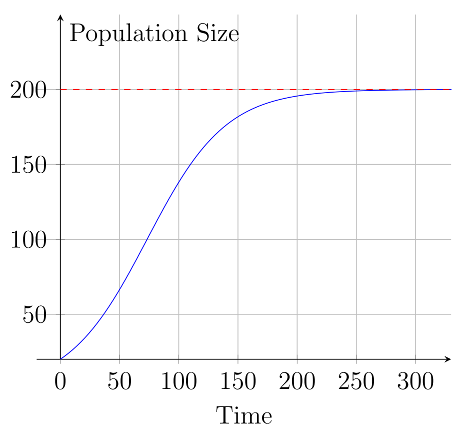

Figure 1.1: Simple logistic model describing population dynamics

Notice that the function asymptotically approaches the line $P = 200$, representing the population's carrying capacity K.

1.3 Stability

Graphically, asymptotes exist at $P = 0$ and $P = K$. These two scenarios occur either when the population is zero, or when the population has reached its carrying capacity K. When the population size P is small, the differential equation is essentially $\frac{dP}{dt} = kP$ since $1 - \frac{P}{K} \approx 1$. In the case where $P > K$, $\frac{dP}{dt} < 0$ implying a decline in population numbers, which would persist until P approaches K, thus making the differential equation approximately equal to zero. This implies that even if we begin with initial conditions where the population size is greater than the carrying capacity, the population in turn will decline and approach asymptote $P = K$ from above. For points between the asymptotes $0 < P(0) < K$, all factors of the differential equation are positive, meaning that the function is strictly increasing. Since all points for $P > 0$ will tend to the asymptotic solution $P = K$, this equilibrium point is known as a sink, meaning all nearby solutions tend to this point. Even with small perturbations, solutions will make their way back toward this value, making it a stable fixed point. Similarly, $P = 0$ is classified as a source since nearby solutions tend away from zero, making it an unstable fixed point.

The plot below depicts solutions for different initial conditions, all which tend to the population's carrying capacity, K, represented by the dotted blue line. Initial population values are $P_0 = 0, 0.01, 2, 6, 12$.

Notice that with even a minuscule perturbation, solutions tend away from $P_0 = 0$.

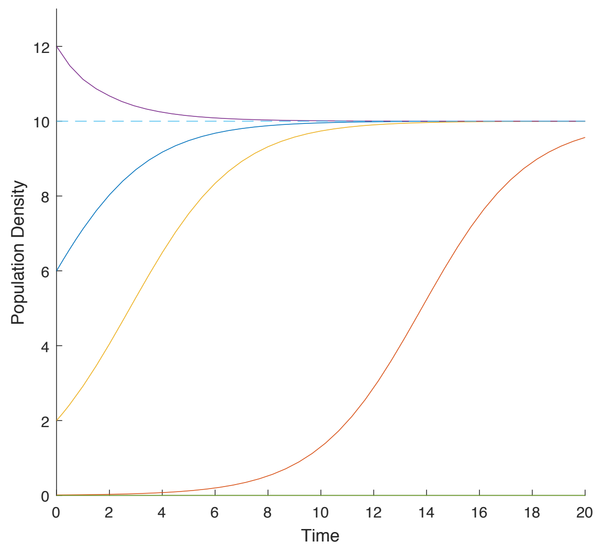

Figure 1.2: Solutions to the logistic equation given different initial conditions

1.4 Model Variations

In the real world, no two distinct species will reproduce, consume resources, and interact with the living environment in exactly the same way. This simple deterministic model may be adjusted by adding in different parameters to better fit the actual dynamics of a certain population. Factors that limit population growth fall into one of two categories:

- Density dependent
- Density independent

Density *dependent* factors limit growth in ways that depend on population density. This means that as a population grows, resources will become limited and the population will approach its ideal size, or carrying capacity. Density-dependent limiting factors may include light, water, nutrients, or disease. Density *independent* factors are characterized as random events having no relationship with population density. Ecologically, these events may include natural disasters such as flash floods and forest fires, as well as human activities such as pesticides and habitat destruction. Typically, many density-dependent and density-independent limiting factors interact to produce the fluctuations and patterns of change that occur in population dynamics.[1]

1.5 Deterministic Models

The populations models discussed thus far are classified as *deterministic models* of population growth. These models predict populations exactly for a given point in time with predetermined parameters. As described, the logistic and exponential growth models are example of continuous deterministic models used to predict the growth rate of populations. Take for example the exponential model. Since the growth rate is directly proportional to the population size, a change in $\frac{dP}{dt}$ with a change in the population size P is consequently linear. This does not model nature very accurately for several reasons. Species may have complex life histories and mating patterns that go unaccounted for, as well as the potential lags in the population growth rate caused by changes in population size. Another assumption embedded in these models is that K, the carrying capacity, is constant over space and time. Environments are continuously changing as time goes on, thus there is no reason to assume the equilibrium number of organisms a given ecosystem can support would be constant.

Chapter 2

Introduction to Stochasticity

2.1 Mathematical and Ecological Significance of Stochasticity

Introducing stochasticity into mathematical models begins to allow their extrapolation into the realm of nature. Chance events occur continuously in nature, and in order to produce a robust mathematical model, such variations and random perturbations are to be accounted for. Unlike deterministic models, stochastic population models incorporate random variations. In deterministic models, the trajectory of the model is wholly determined by initial conditions and its parameter values. Stochastic models involve some inherent amount of randomness, potentially causing near identical inputs to produce drastically varied outputs. Fluctuations in populations are often caused by demographic and environmental stochasticity, rather than by chaos or internally generated cycles. The inherent heterogeneity of populations are likely to cause strongly stochastic effects on the system's dynamics in the natural world. Stochastic differential equation models are among the basic population models that incorporate demographic and environmental stochasticity. Stochasticity is important in population growth models, and must be incorporated in order to better understand the nature of population fluctuations. Knowledge of both demographic and environmental stochasticity is essential for explaining the temporal changes in populations and finding approximations for probabilities of extinctions as well as expected times to extinction, which will be explored in Chapter 4.

Demographic stochasticity results from random independent events affecting individual mortality and reproduction rates, causing random fluctuations in net population growth rate with a more drastic impact on small populations. *Environmental stochasticity* results from ephemeral fluctuations in mortality and reproductive rates of an entire population, causing a population's growth rate to fluctuate randomly in populations of all sizes. [2]

A simple example of *demographic* stochasticity would be if during one year, a specific fish species declines in numbers due to a chance production of fewer eggs among individual members of the population. These sort of fluctuations vary at the individual level, and most strongly impact small populations.

Environmental stochasticity would better describe the population dynamics of the Agave plant. This species of plant, reproduces only once during its lengthy lifespan. Average lifespans are about 25 years, though individual lifespans vary depending on irregular rainfall. Only after receiving enough rain to allow sufficient growth will Agaves reproduce. Eventually, an exceptionally wet season occurs and the plants will flower, produce a large number of seeds, then die. This way, even in areas where the amount of rainfall is scarce and unpredictable, the plants are able to gather sufficient water before reproducing. Essentially, the growth and eventual reproduction of this species is dependent on something unpredictable and erratic. The density-independent factor rainfall is unrelated to population density, but merely an environmental fluctuation affecting all individuals in more or less the same way. The

density-independent factor rainfall limits birth rate, which in turn limits growth rate, but due to its unpredictability, cannot regulate Agave populations.
Today, with the climate changing all over the world, more and more of these effects will impact the population dynamics of nearly every species known and unknown. Some environmental variations include local weather changes, variation in individual health, and deforestation or forest fires that kill off large numbers of individuals.

2.2 Random Walks

In the realm of population dynamics, it is crucial to model the path taken in order to explain past trends and accurately predict the future. Nature is inundated with exceedingly complex processes, such as growth, which may be difficult to approximate. The natural world is also inherently subject to stochasticity.
A *random walk* is an example of a Markov process, where future behavior is independent of past history. The random walk model describes the process of determining the location of a point taking equal sized steps in arbitrary directions, up or down for example, with equal probability. According to this scenario infamously titled the "drunkard's walk" problem, the distance a drunken person travels while making random left and right turns is equal to their typical step size times the square root of the number of steps taken. [3]

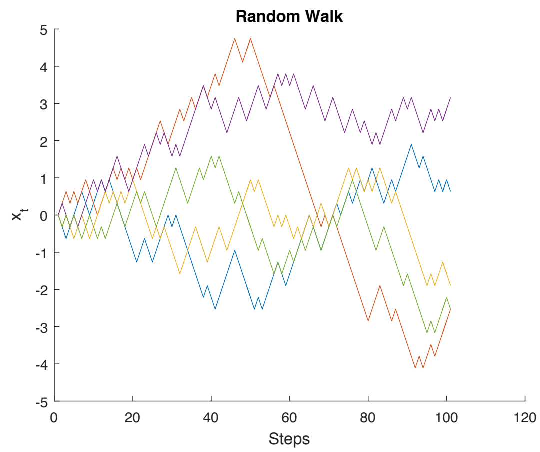

Figure 2.1: Five random walks

Figure 2.1 was generated with the following code with $n = 100$:

```
function [ t ] = myRandomWalk( n )

 N = n;  % number of steps
 dt = 0.1;
 sdt = sqrt(dt);
 x_t(1) = 0;
     for n = 1:N
          a = sign(randn)*sdt;
          x_t(n+1) = x_t(n) + a;
     end
plot(x_t)
xlabel('Steps'), ylabel('x_t'), title('Random Walk')

end
```

2.3 Brownian Motion

When you take discrete random walks with definite time steps and continually decrease the step size, the process tends toward continuity. Continuous Brownian motion is a random walk in minuscule steps. Particularly when observed for extended time scales, stochastic processes behave like random walks with small but frequent jumps. The continuous-time stochastic process $W_n(t)$ with $t \geqslant 0$ is a random process that describes the evolution of a system over time. The number of steps taken is represented by n, and t serves as units of time. For example, after we have taken n steps, we have

$$W_n(1) = \sum_{i=1}^{n} X_i \sqrt{\frac{1}{n}}$$

for random variable X_i. We describe X_i as taking on a value of either -1 or 1 with probability $p = \frac{1}{2}$, producing a symmetric "walk". The length of our time steps is set as $\Delta t = \frac{1}{n}$. As Brownian motion progresses, behavior at each increment of time is independent of past behavior. Therefore, the best predictor for the future is based on current behavior, classifying Brownian motion as a Martingale. Since the random variables are independent, the *expected value* of independent increments of $W_n(t)$ is simply the average of values.

The expected value for our symmetric set-up becomes zero, as seen below:

$$\mathbf{E}[W_n(1)] = \mathbf{E}\left[\sum_{i=1}^{n} X_i \sqrt{\frac{1}{n}}\right] = \sum_{i=1}^{n} \sqrt{\frac{1}{n}}\, \mathbf{E}[X_i] = \sum_{i=1}^{n} \sqrt{\frac{1}{n}} \cdot 0 = 0$$

since the average or expectation of X_i values is 0. This can be seen in the figure below, with 100 paths and 100 time-steps:

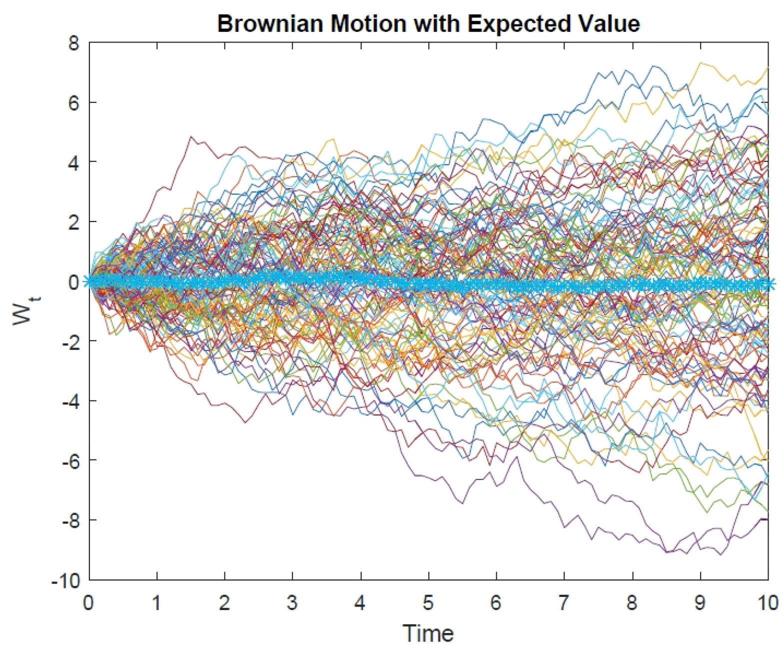

Figure 2.2: Brownian motion plotted with expected value

In order to measure how far the set of random numbers is spread out from its average or expected value, we calculate the variance:

$$Var[W_n(1)] = Var\left[\sum_{i=1}^{n} X_i\sqrt{\frac{1}{n}}\right] = \frac{1}{n}\sum_{i=1}^{n} Var[X_i] = \frac{1}{n}\sum_{i=1}^{n} 1 = \frac{1}{n}\cdot n = 1$$

which turns out to be dependent on the interval size, increasing proportionally with time. For large finite n, we can approximate the distribution of $W_n(1)$ by a normal distribution using the Central Limit Theorem: $W_n(1) \sim \mathcal{N}(0,\, 1)$. These calculations extend to any t for $W_n(t)$, thus implying $W_n(t) \sim \mathcal{N}(0,\, t)$ since $Var[W_n(t)] = t$. [4]

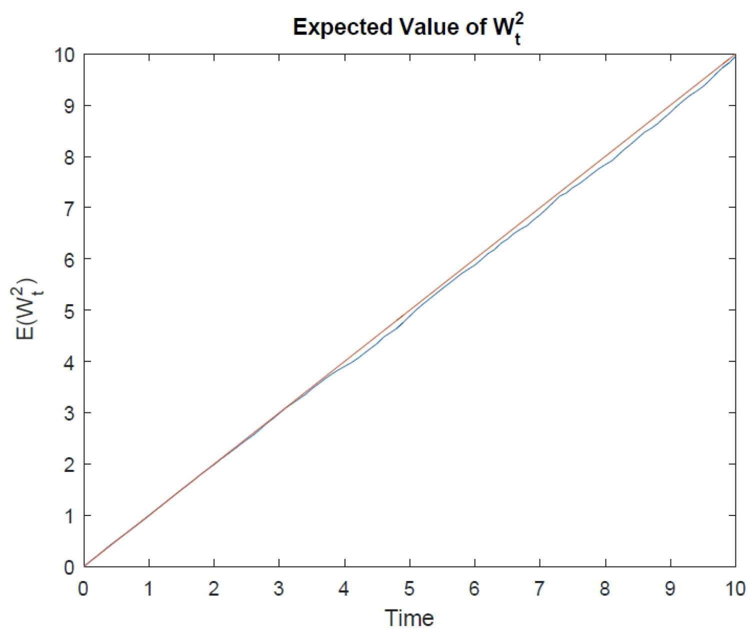

Figure 2.3: Expected value of W_t^2 compared to $f(t) = t$

The following graphs reveal the transformation from discrete random walks into continuous Brownian motion. Continuity emerges as the step size is made infinitely small, and additionally the expected value will not change, as it is independent of n. Depicted are four runs with 8 walks each with 10, 100, 1000, and 100,000 time-steps.

Figure 2.4: Convergence to Brownian motion

If each individual step is random, what can be predicted about the probability distribution function of the distance of each point from the origin? The Central Limit Theorem affirms that, with a sufficient number of independent identically distributed random variables, the means of samples tend to a normal distribution. Regardless of the shape of the actual distribution, this allows us to approximate our distribution with a large enough sample size. The following histograms in Figure 2.5 compare final distance traveled from the origin with the Gaussian distribution, represented by the red curve. The data was produced with a code simulating Brownian motion with an initial value of 0 for 100 steps, with 10 steps per unit of time. Runs generated with sample size = 50, 500, 5000, and 50,000 in respective order.

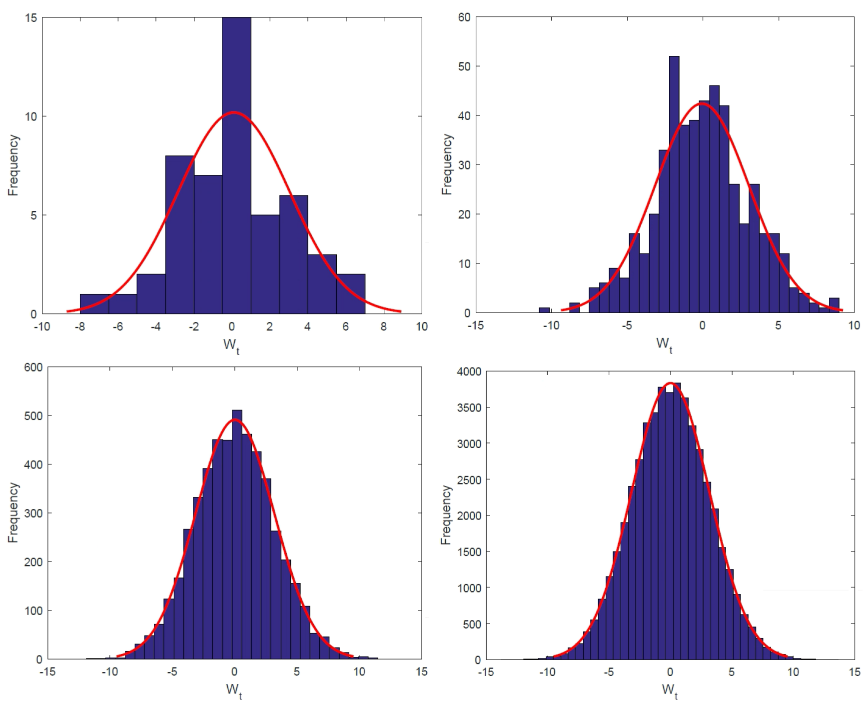

Figure 2.5: Resulting histograms comparing the distribution of Brownian motion path destinations to a normal distribution

2.4 Itô's Lemma

Assume the process X_t satisfies the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dW_t$$

where $\mu_t = \mu(X_t, t)$ and $\sigma_t = \sigma(X_t, t)$.
Also note that $dW_t = \sqrt{dt}$, or equivalently $(dW_t)^2 = dt$. This can be roughly justified by computing the expected value of $(W_{t+\Delta t} - W_t)^2$:

$$\mathbf{E}[(W_{t+\Delta t} - W_t)^2] = Var(W_{t+\Delta t} - W_t) = \Delta t$$

This is due to the property of Brownian motion stating $W_{t+s} - W_t$ has variance s since W_t are independent and identically distributed.
In theory, we set $\Delta W = \pm\sqrt{\Delta t}$ with $P = \frac{1}{2}$, but for numerical purposes, $\Delta W = r\sqrt{\Delta t}$ with $r \sim \mathcal{N}(0, 1)$. It follows that a change in $(dW_t)^2$ is caused by a change in the parameter dt, making it determinant and not random, with a magnitude of dt.
In the case of only constant deterministic growth, $\sigma(X_t) = 0$, making our equation

$$dX_t = \mu(X_t)dt$$

Now, if $Y_t = f(X_t)$, the equation describing Y_t would be

$$dY_t = f'(X_t)dX_t = f'(X_t)\mu(X_t)dt$$

and by the chain rule,

$$\begin{aligned}\frac{dY_t}{dt} &= f'(X_t)\frac{dX_t}{dt} \\ &= f'(X_t)\mu(X_t)\end{aligned}$$

However, in the instance where f depends on a real variable t as well as on a stochastic process of Brownian motion W_t, we will heuristically derive by calculating the second order Taylor Expansion of f about X_t:

$$\begin{aligned}dY_t &= f(X_t + dX_t) - f(X_t) \\ &= f(X_t) + f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 - f(X_t) \\ &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)[\mu(X_t)dt + \sigma(X_t)dW_t]^2 \\ &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)[\mu^2(dt)^2 + 2\mu\sigma dt dW_t + \sigma^2(dW_t)^2]\end{aligned}$$

Since the limit of $dt \to 0$, the terms $\ll dt$ are dropped, with dX_t^2 given by: $dt^2 = 0, dtdW_t = 0$ and $dW_t^2 = dt$.
Thus our equation simplifies to

$$dY_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)\sigma^2 dt \tag{2.1}$$

$$= [f'(X_t)\mu(X_t) + \frac{1}{2}f''(X_t)\sigma^2(X_t)]dt + f'(X_t)\sigma(X_t)dW_t \tag{2.2}$$

The term $\frac{1}{2}f''(X_t)\sigma^2 dt$ in Equation (2.1) is known as our "Itô Term" which stems from this generalization of the chain rule in the stochastic setting. Equation (2.2) is called *Itô's Lemma*, and the equation has been rewritten to separate the deterministic terms from the stochastic terms. Itô's Lemma serves as the stochastic calculus counterpart of the chain rule, making possible the computation to find the differential of a time-dependent function of a stochastic process such as Brownian motion.

The reason for computing the second order Taylor Expansion is due to Brownian motion's infinite linear variation. However, since it has paths with finite quadratic variation, this approximation is sufficient.

Now we apply Itô's Lemma to find the solution to the exponential function $f(x) = e^x$ with $f(x) = f'(x) = f''(x)$.

For the stochastic process

$$X_t = \sigma_t W_t + \mu_t$$

with constant σ and μ, we define the process $Y_t = f(X_t) = e^{X_t} = e^{\sigma W_t + \mu t}$ as Exponential Brownian motion. By Itô's Lemma,

$$dY_t = Y_t(\sigma dW_t + (\mu + \frac{1}{2}\sigma^2)dt)$$

For the case where $\mu = -\frac{\sigma^2}{2}$:

$$Y_t = e^{\sigma W_t - \frac{\sigma^2}{2}t}$$

which implies

$$dY_t = Y_t \sigma dW_t$$

Chapter 3

Numerical Methods in MATLAB

3.1 Solving SDEs arising in population dynamics

Examples utilize MATLAB's built-in *randn* function, which generates a normally distributed, randomly selected floating point number. These numbers are intended to be independent samples from the normal distribution $\mathcal{N}(0, 1)$.
Consider the SDE

$$dX_t = \mu_t(X_t, t)dt + \sigma(X_t, t)dW_t$$

Given an SDE depending on t, we add the deterministic component to the random component. The issue is how to quantify the Brownian increment dW_t. As seen through our analysis of Brownian motion, we draw a random number r from $\mathcal{N}(0, 1)$ and multiply it by $\sqrt{\Delta t}$ in order to achieve convergence, thereby replacing dW_t by $r\sqrt{\Delta t}$ in our calculations.

3.2 Comparing analytical and numerical solutions

In order to further justify our numerics, we look to compare expectations for numerical and analytical solutions. For the probability density function

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

and expectation

$$\mathbf{E}(e^X) = \int_{\mathbb{R}} p(x)e^x dx$$

in order to compute $\mathbf{E}(e^X)$ we look to solve the following integral:

$$\int_{\mathbb{R}} e^x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

By algebra and completing the square, this becomes

$$e^{\mu+\frac{\sigma^2}{2}} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(\tilde{x}-\sigma^2)^2} d\tilde{x}$$

with the substitution $\tilde{x} = x - \mu$ and $d\tilde{x} = dx$. By the Gaussian integral

$$I = \int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}$$

then squaring and converting to polar coordinates yields

$$\begin{aligned} I^2 &= \iint_{\mathbb{R}} e^{-a(x^2+y^2)} dxdy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r dr d\theta \end{aligned}$$

with $a = \frac{1}{2\sigma^2}$. Evaluating at our limits produces

$$I^2 = \frac{\pi}{a} \Rightarrow I = \sqrt{\frac{\pi}{a}}$$

and therefore

$$\begin{aligned}\mathbf{E}(e^X) &= e^{\mu+\frac{\sigma^2}{2}}\left[\frac{1}{\sqrt{2\pi\sigma^2}}\sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}}\right] \\ &= e^{\mu+\frac{\sigma^2}{2}}\left[1\right] \\ &= e^{\mu+\frac{\sigma^2}{2}} \qquad (3.1)\end{aligned}$$

Essentially, this is a normal distribution with mean σ^2 and variance σ^2 and our Expectation $E(e^X) = 1$ due to the normalization factor. Thus, Equation (3.1) has been derived as our analytical formula for computing expectation.

3.3 Calculating Passage Time of Brownian Motion Trajectories

Events are often triggered when a stochastic or random process first encounters a threshold. The threshold can be a barrier, boundary or specified state of a system. The amount of time required for a stochastic process, starting from some initial state, to encounter a threshold for the first time is often referred to as the first hitting time. In statistics, first-hitting-time models are a sub-class of survival models. The first hitting time, also called first passage time, of a trajectory with respect to an instance of a stochastic process is the time until the stochastic process first falls below a specified threshold. Given that Brownian motion is used often as a tool to understand and model more complex phenomena, it is important to understand the probability of a first passage time for a Brownian motion trajectory in reaching some threshold value.[5] A particle in Brownian motion taking the time to hit a specific point a may be modeled by this distribution by setting its scale parameter $c = a^2$. The probability density function of the Lévy Distribution is

$$p(x) = \sqrt{\frac{c}{2\pi}} \frac{e^{-c/2(x-a)}}{(x-a)^{3/2}}$$

for scale parameter c with $c > 0$ and location parameter a with $x \geqslant a$. As previously discussed, the formula for computing expectation is

$$\mathbf{E}(x) = \int_0^\infty xp(x)dx$$

For $a = 0$, the *first moment*, or expectation, of the Lévy Distribution is defined as

$$\sqrt{\frac{c}{2\pi}} \int_0^\infty \frac{xe^{-c/2x}}{x^{3/2}}dx$$

which diverges as $x \to \infty$.

This is investigated through the following means. With a threshold value specified as $K = 2$, numerous Brownian motion paths were computed, and the times for each path to hit this threshold were stored in a vector t_fall. The following histograms in Figure 3.1 depict hitting times for these trajectories with 1000 time-steps, increasing the number of runs to reveal convergence, if any.

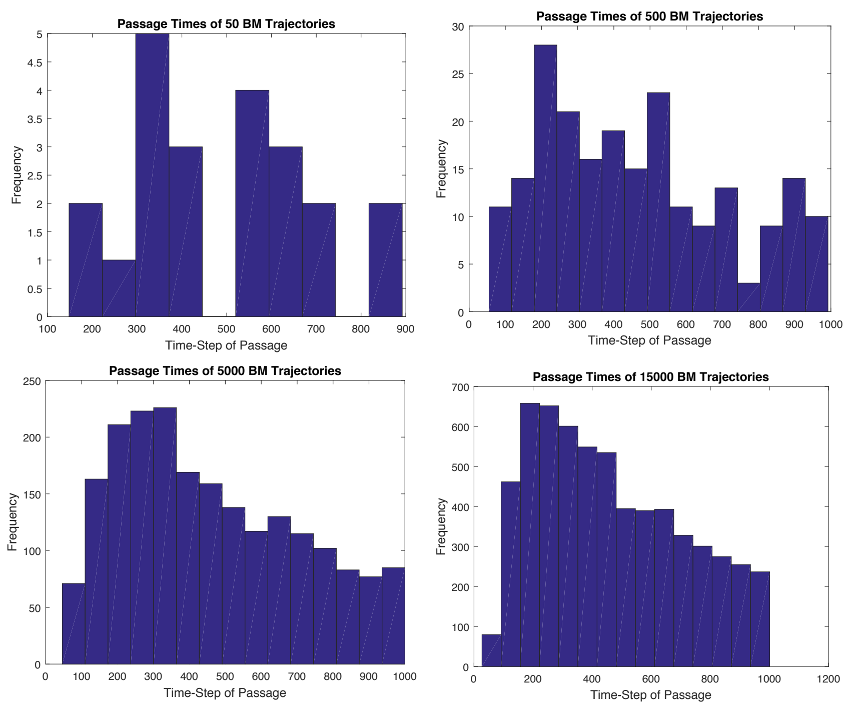

Figure 3.1: Resulting histograms showing first passage times of Brownian motion trajectories

Note: There is no convergence to a Gaussian Distribution. Instead, this data follows what is known as a Lévy Distribution. These distributions have heavy tails, which contributes to the fact that the expectation diverges. The typical passage time for these distributions can instead be estimated by the maximum value.

Chapter 4

Applications and Explorations

4.1 Quantifying Time to Extinction

A process is said to be ecologically stable if the extinction does not occur within a realizable time interval. A quantity of interest is the expected time until extinction, which we will call E_x. After running our simulation to output stochastic population trajectories using Exponential Brownian motion, we can calculate the average time for trajectories to fall below a threshold value, functionally sending populations to extinction. We also calculate the percentage of species to go extinct during the specified time interval with identical initial conditions.

As previously mentioned, the distribution of Brownian motion hitting times has infinite expectation. For the Lévy distribution, typical hitting times will be estimated using the maximum on the distribution.

One simulation of our Exponential Brownian motion model applied to population dynamics, *mystochpops.m*, with 300 time-steps and 200 paths produces:

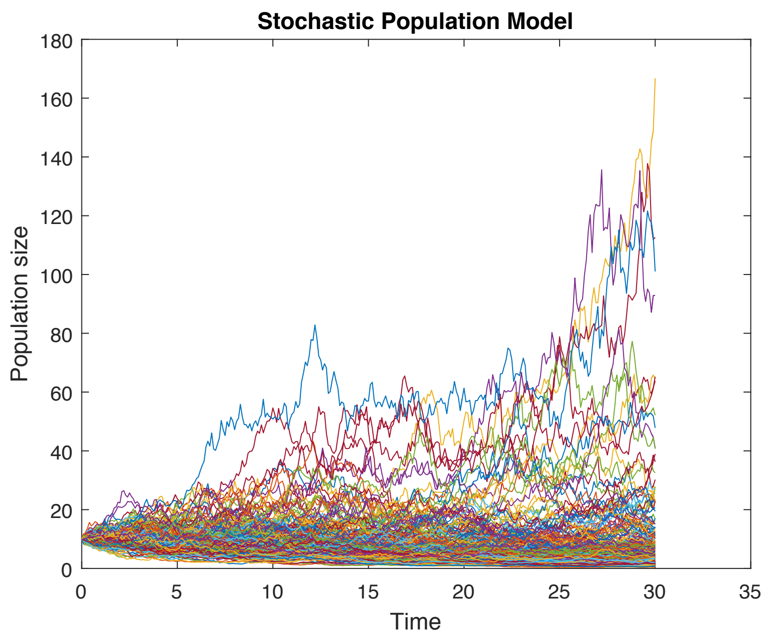

Figure 4.1: Exponential Brownian motion simulating population dynamics

With the threshold value set at 2 and initial population size of 10, the typical extinction time for this single simulation is at time-step 206.8431, or time $t = 20.684$, and the proportion of species that reach extinction $P(ext) = 0.2550$.

4.2 Stochastic Logistic Models

Stochasticity is incorporated into the deterministic logistic population model with the following techniques using the Forward Euler Scheme. There are numerous approaches to be investigated involving how to incorporate the noise term. In this first model, a linear noise term $\sigma * randn * \sqrt{dt}$ is added on at each time-step. Graphics below for carrying capacity $K = 5000$, per capita growth rate $r = 0.1$, $P_0 = 100$, and varying σ. We assign $\sigma = 0, 30, 50, 70$ from top left to bottom right, respectively.

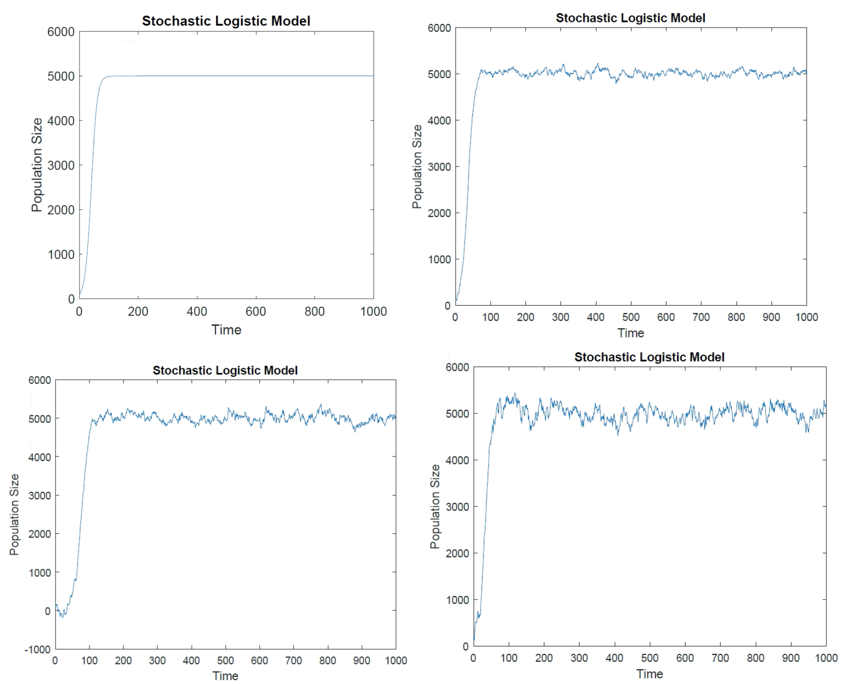

Figure 4.2: Stochastic Logistic model with varied degrees of added noise

Notice that this additional noise term does not drastically alter the trajectory. Slight perturbations continue to repel the path from the initial state, and the carrying capacity value will continually attract the function despite perturbations.

The next model to explore involves a fluctuating carrying capacity. At each time-step, our stochastic term is added onto the carrying capacity, causing the function's equilibrium to wander randomly. This added stochasticity simulates the continuously changing capacity of an ecosystem. Below are four realizations with initial carrying capacity $K = 5000$, per capita growth rate $r = 0.5$, $P_0 = 500$, and varying σ. We assign $\sigma = 10, 50, 80, 200$ from top left to bottom right, respectively.

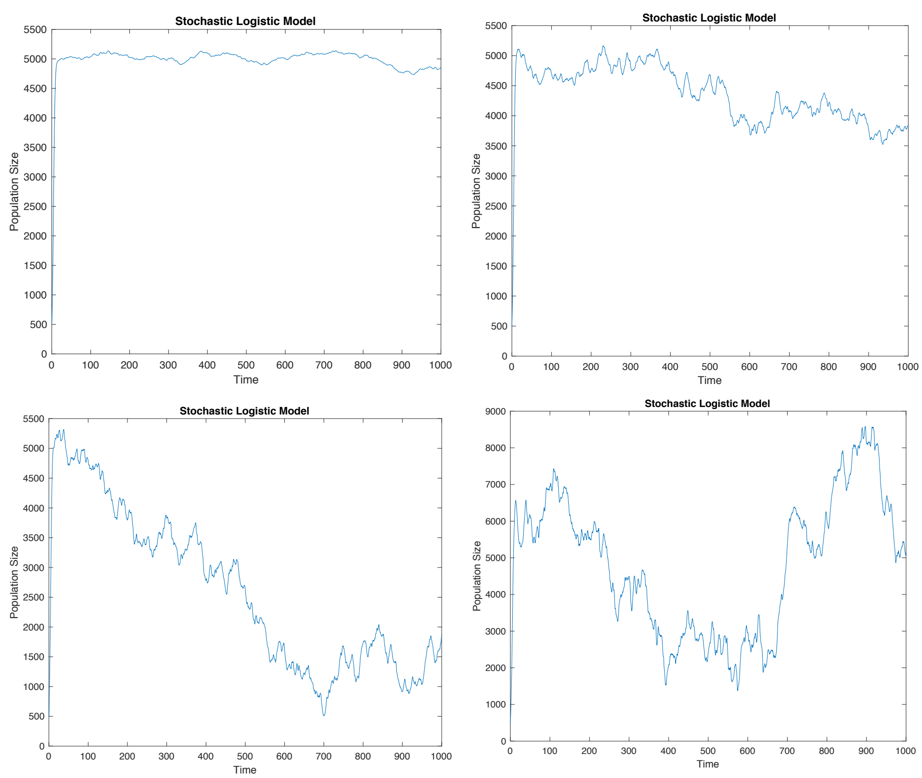

Figure 4.3: Stochastic Logistic model with fluctuating carrying capacity

Chapter 5

Conclusions

5.1 Future Research

Future research should delve into which variable(s) to put the stochastic term into, and how exactly to incorporate it. We may bring stochasticity into the growth rate, the carrying capacity, or an added on linear noise term. Each of these methods will produce a different answer. Inserting stochasticity into a fixed term would mean that near either fixed point, there would be little effect. Inserting stochasticity into the carrying capacity moves around the system's stable fixed point. Thus, trajectories would constantly trying to reach the new and continuously changing fixed point. In this proposed case, would it still be correct to classify this "stable fixed point" as stable or fixed? Ecologically, this makes sense due to an increase in environmental stochasticity arising from a changing climate. Varying weather, food supply and loss of habitat are key components that constitute what the environment can support, i.e. the carrying capacity. An added linear term of Brownian motion with its nice convergence theorems can be handled analytically, which is one reason it is a significant simple mathematical model for describing and exploring how to handle stochasticity.

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