OPTION PRICING: FOREIGN EXCHANGE AND QUANTOS

HONORS THESIS
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I certify that I have read this honors thesis and that, in my opinion, it is fully adequate in scope and quality as an honors thesis for the degree of Bachelor of Science in Mathematics.

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Chapter 1

Introduction: Financial Markets

Much of which drives the present day global economy is the existence of financial markets where percent ownership of corporations are traded via stocks. There are many factors that influence the price of a financial asset and it is often unclear what exactly drives the changes of a value of a stock at a particular moment in time. Therefore, it is difficult to determine which stock is worth purchasing. Many financial institutions have the aim to make a profit based on the stock market without accruing any of the risk that comes along with "betting" on stocks or, at least they intend to keep their risk a a minimum level. One way of achieving is via buying and selling particular financial instruments, such as derivatives and options. In the following, we will introduce several examples of such financial contracts.

1.1 Options

An option is a contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a specific price on or before a certain date. If the option gives the holder the right to buy, the option is called an *call option*. If the option gives the right to sell, the option is called a *put option*. Options also differ concerning the rule when the right to buy or sell can be exercised: An option that can only be exercised at maturity is called an European option, if the option can be exercised at any time (until it matures), the option is called an Americal Option. An option, just like a stock or bond, is a security. It is also a binding contract with strictly defined terms and properties.

1.2 Derivatives

A derivative is a more general term for a contract that derives its' value from the performance of an underlying entity.This underlying entity can be an asset, index, or interest rate, and is often called the "underlying". Options are examples of derivatives and the "underlying" is the corresponding stock. Other common examples are *futures* and *forwards*. Later in this work, we will discuss *quantos* which are derivatives involving stocks on foreign markets. An important task is to price derivatives. This is commonly done with tools developed in the context of stochastic differential equations. The next section presents a short introduction to derivative pricing.

Chapter 2

Financial Mathematics

A main objective of financial institutions when selling options or similar financial instruments is to make a profit regardless of how the stock price changes by taking a commission and remaining in a risk-neutral position. An important example is the Black-Scholes model, leading to the Black-Scholes equation describing the risk-free value of an European call option if the underlying stock is assumed to follow a certain stochastic process called exponential Brownian motion.

We will follow Martin Baxter's and Andrew Rennie's definition of a random walk and eventually Brownian Motion from their book: **Financial Calculus** [1]. Let's begin by defining a random walk which is a basic example of a *discrete stochastic process*:

2.1 The random walk

Denoted as W_n(t) with the following properties:

1. $W_n(0) = 0$,

2. Layer spacing $1/n$,

3. Up and down jumps equal and of size $1/\sqrt{n}$,

4. A measure $\mathbb{P}$, given by up and down probabilities everywhere equal to $\frac{1}{2}$.

Using the central limit theorem, they show that as n gets large, $W_n(1)$'s distribution tends to be normally distributed with mean equal to zero and standard deviation equal to one. Moreover, they show that the random variable $W_n(t)$ can be written in terms of simple random variables X_i which take the value 1 or -1 with probability 1/2:

$$W_n(t) = \sqrt{t}\left(\frac{\sum_{n=1}^{nt} X_i}{\sqrt{nt}}\right)$$

with the distribution of the ratio tending to N(0,1) and thus W_n(t) tends to normal N(0,t).

It is important to note that each random walk has the property that its future movements are independent of what has already occurred. Therefore, again by the central limit theorem, we see that $W_n(s+t) - W_n(s)$ tends to the same distribution of N(0,t) as the original random walk. It is then stated that the distribution of W_n converges in the large-n limit, and it converges towards Brownian Motion, which is a *continuous stochastic process*. Brownian Motion has the following properties:

2.2 Brownian Motion

1. W_t is continuous, and $W_0 = 0$,

2. the value of W_t is distributed, under $\mathbb{P}$, as a normal random variable N(0,t),

3. the increment $W_{s+t} - W_s$ is distributed as normal N(0,t), under $\mathbb{P}$, and is independent of $\mathcal{F}_s$ the history of what the process did up to time s.

It is important to note, that W is continuous everywhere but differentiable nowhere.

With the definition of Brownian Motion, we can define our simple stock model for an asset X_t as exponential Brownian motion, namely:

$$X_t = X_0 \exp(\sigma W_t + \mu t)$$

with the following properties:

1. We use an exponential funciton due to the fact that stock prices can never go negative.
2. We add a "drift" term μ to account for long term growth of inflation at the very least. In the simplest model, the drift μ is assumed to be constant.
3. We multiply out Brownian Motion term by a factor of σ to scale correctly. The constant σ is called *volatility* of the stock.

Now that we have a simple stock model, we can begin to investigate how we can analyze it.

2.3 Stochastic Calculus

Stock behavior is modeled by Brownian motion and must be treated with different tools than functions that are differentiable. When differentiating a smooth function, we write the change in a value over a time interval as

$$df_t = \mu_t dt$$

with the function μ_t as the slope or drift.

In many models, not only in finance but also in other fields like physics, the drift μ_t can depend on the current value of the function, and thus it is $\mu(f_t, t)$, where $\mu(x, t)$ is a known function, then we write df_t as an ordinary differential equation,

$$df_t = \mu(f_t, t)dt$$

In the stochastic world, we can take a similar approach, but we need to account for both random and deterministic changes. Consider therefore changes of a stochastic process X_t that has both a Newtonian term based on dt and a Brownian term, based on an infinitesimal increment of Brownian motion W_t. For this stochastic process X_t, we can see that the infinitesimal change of X_t is given by

$$dX_t = \sigma_t dW_t + \mu_t dt$$

It should be noted that μ_t can depend on time t and also be random and depend on values of X_t up until time t (along with σ_t, which is known as the volatility of X_t at time t). Formally, we can represent the stochastic process X_t for $t \geqslant 0$ as a stochastic integral:

$$X_t = X_0 + \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds$$

2.4 Ito's lemma

Since we are working with stochastic processes, we need to use a slight variation of calculus, known as Ito calculus.

Ito's formula

If X is a stochastic process, satisfying $dX_t = \sigma_t dW_t + \mu_t dt$, and f is a deterministic twice continuously differentiable function, then $Y_t := f(X_t)$ is also a stochastic process and is given by

$$dY_t = (\sigma_t f'(X_t))dW_t + (\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t))dt$$

We can consider Ito's lemma as an expansion of a Taylor series up to the second term. Essentially, it is the stochastic version of the chain rule:

$$df = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t$$

and now we can begin the derivation for the Black-Scholes equation.

2.5 Black-Scholes Equation

We can define a simple first model where interest rates are compounded continuously and the stock follows exponential Brownian motion:

$$B_t = \exp(rt), \qquad S_t = S_0 \exp(\sigma W_t + \mu t) .$$

Here, the constant r is the riskless interest rate, σ is the stock volatility and μ is the stock drift. There are no transaction costs and both instruments are freely and instantaneously tradable either long or short at the price quoted. This allows us to build a model that consists of a riskless constant-interest rate cash bond and a stock following the above exponential Brownian motion with volatility σ.

Since the stock follows an exponential Brownian motion $S_t = \exp(\sigma W_t + \mu t)$, the logarithm of the stock price, $Y_t = \log(S_t)$, follows a simple drifting Brownian motion $Y_t = \sigma W_t + \mu t$. Thus the SDE for Y_t is easy to write down: $dY_t = \sigma dW_t + \mu dt$. But, of course, Ito makes it possible to write down the SDE for $S_t = \exp(Y_t)$ as

$$dS_t = \sigma S_t dW_t + \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt\,.$$

By the Cameron-Martin-Girsanov theorem, there exists a measure $\mathbb{Q}$ such that S_t is a $\mathbb{Q}$ martingale, and, under this new measure, the SDE becomes:

$$dS_t = \sigma S_t d\tilde{W}_t$$

Here, $\tilde{W}_t$ is an exponential Brownian motion under $\mathbb{Q}$, and by solving the above SDE we see that we can write the stock process S_t in terms of this Brownian motion $\tilde{W}_t$ as

$$S_t = S_0 \exp\left(\sigma\tilde{W}_t - \frac{1}{2}\sigma^2 t\right)$$

f In the case of zero interest rates, the value of a European call option V with strike price K is given by

$$V = \mathbb{E}_{\mathbb{Q}}((S_T - k)^+)$$

After accounting for an interest rate $r > 0$ and evaluating this expectation using the probability density of Brownian motion, we obtain the **Black-Scholes formula** for pricing European call options.

$$V(s,T) = s\Phi\Big(\frac{\log\frac{s}{k} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\Big) - k\mathrm{e}^{-rT}\Phi\Big(\frac{\log\frac{s}{k} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\Big)$$

Here, s is the initial stock price, k the strike price, T the maturity, and σ the volatility of the stock.

Lets quickly reiterate the basis of our model. The price of the underlying stock follows a geometric Brownian motion. This is where the Brownian motion W_t models the random changes. Intuitively, W_t is a process that "jumps" up and down in a random way and its expected value over any time interval is 0. Also, its variance over time is equal to T, we can think of W as a "random walk". Therefore, the infinitesimal rate of return on the stock has an expected value of $\mu\, dt$ and a variance of $\sigma^2 dt$.

The payoff of an option $V(S,T)$ at maturity is known. To find its value at an earlier time we need to know how V evolves as a function of S and t. By Ito's lemma for two variables we have (for the Black-Scholes model) the change of the value of the portfolio V as

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S}\, dW$$

Chapter 3

Monte-Carlo Simulations

Note: All code in the following section can be assumed to be used with Matlab. We begin by testing the Black-Scholes formula using Monte-Carlo simulations.

3.1 Pricing an option in Matlab

Input needed: a mean, standard deviation, end time, vector spacings, and how many random walks you would like to compute - m,s,e,k,p respectively

Code for Monte-Carlo option pricing:

```
function [t,f,g,h] = Ito(m,s,e,k,p)

  t = linspace(0,e,k);          % set domain for our function
  Wt = zeros(p,k);
  f = zeros(p,k);
  g = zeros(p,1);               % to take end values of f
  h = zeros(1,p);               % for change to a row vector
```

```
  dt = t(2) - t(1);              % set infinitesimal values for derivative
  sdt = sqrt(dt);                % needed for Ito's formula
  f(:,1) = [50];                 % starting stock price of 50
  for i = 2 : k
     Wt(:,i) = Wt(:,i-1) + randn(p,1) * sdt;       % filling Wt: random values
     f(:,i) = 50*(exp(s*Wt(:,i)-0.5*s^2*[t(i)]));  % Black-Scholes formula
  end

  g(:,1) = max(f(:,k) - 55 ,0); % only values greater than strike price
  h(1,:) = g(:,1);               % use mean(h) to compute value
end
```

The code creates an ensemble of p Brownian paths using the risk-neutral measure. Then these paths can be used to construct the corresponding stock paths. We can use this program to visualize both Brownian motion and a stock following exponential Brownian motion. Figure 3.1 shows a single path: A single path, however, does not allow us to check easily whether our algorithm is correct. A good first check is to look at a large ensemble of paths and the distribution of their values. In particular, the distribution of the end values should be close to a normal distribution curve. The following figure 3.2 shows the full ensemble of 5000 paths.

The parameters entered were $m = 0$, $s = 0.3$, $e = 2$, $k = 200$, $p = 5000$. At first glance, the picture confirms our intuition: The majority of the end values are clustered near the middle, with few walks deviating significantly.

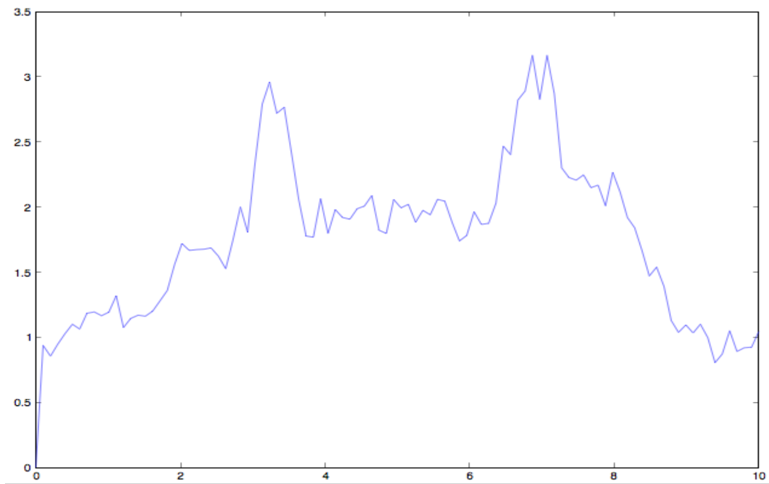

Figure 3.1: Example of a Brownian path created using Matlab.

3.2 Checking the distribution with a bar graph

We will be comparing our results on a histogram to check our distribution. We cannot expect it to look perfect as the bucket sizes will affect it. We will be using Matlab's hist() command to produce our graph.
As we can see from the figure below, the values closest to our mean were the most likely to occur, with the likely-hood decreasing as the deviation increases. Now that we have confidence in our model, lets run a simulation to check the accuracy.

We will run a test with parameters $m = 0$, $s = 0.3$, $e = 2$, $k = 800$, $p = 100000$, stock price equals 100, and strike price equals 102. With these values, our Monte-Carlo simulation yields as a result for the option price 16.040.

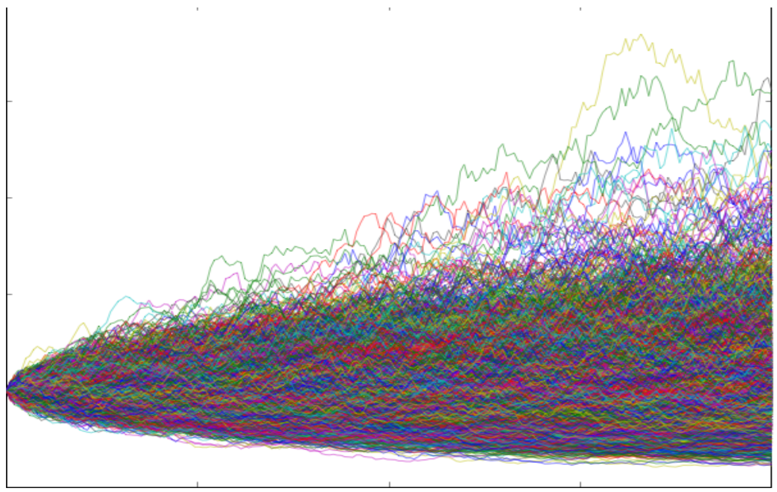

Figure 3.2: Distribution of the end values for an ensemble of Brownian paths.

3.3 Comparison with Black-Scholes

To check our result, we will code the explicit Black-Scholes formula from above, and see if we get similar results. Something to notice is that the Black-Scholes model has an interest rate r that we cannot ignore. Since our first model did not take interest rates into account, any time this formula is used the interest rate r will be set to zero.

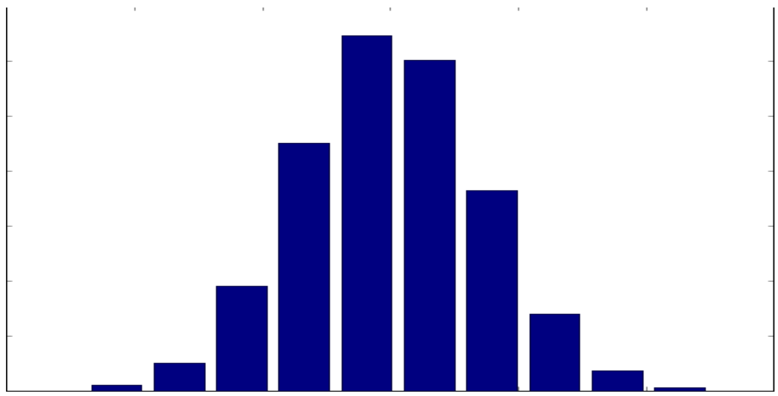

Figure 3.3: Bar graph with end values of 5000 random walks.

Code for analytical Black-Scholes option pricing:

```
function price= blackScholes(Stock,Strike, r, s, T)

dt = sqrt(T);

d1 = (log(Stock/Strike)+r*T)/(s*dt)+0.5*s*dt;
d2 = d1-(sigma*dt);

price  = Stock*normcdf(d1)-Strike*exp(-r*T)*normcdf(d2)
end
```

Plugging in Stock = 100, Strike = 102, r = 0.0001, s = 0.3, T = 2 – our result from the analytical formula above is 15.997. The results are close. We cannot expect them to be exactly the same since our first model relied so heavily on random numbers - if we ran our first simulation hundreds of times, we would assume that the value would average very close to the same value that the Black-Scholes formula gives us.

3.4 Our Model and the Real World

It's nice to see that our model coincided with Black-Schaoles so well, but we must check its applicability in the real world. As we already have noted, our model does not take interest rates into account, thus we already have a problem translating to reality. Interest rates, however, can be easily incorporated by an additional exponential term. We must also note that our simulation was accurate for the formula to price European call options – we have not attempted to price put options nor American options all together. While the code for the pricing of a put option can be obtained by a simple change of the claim, pricing of American options is much more challenging. Option pricing in the real world also takes into account if the stock pays dividends and all of this must be accounted for.

Chapter 4

Foreign Exchange

The derivative market for the foreign exchange stems from the varying value of the US dollar compared with other currencies. We must handle these situations with a little caution, as these instruments behave differently, leading to some subtleties. For example, in our stock model, we can see a basic forward price as $S_0 e^{rT}$. This result can be obtained by simple no-arbitrage argument. When trying to extend this argument to the foreign exchange, we must keep in mind that there will be cash bonds in both currencies along with an exchange rate.

4.1 Black-Scholes Currency Model

Before investigating Quantos, lets first look into how Black-Scholes handles the exchange. We will define two cash bonds and the exchange rate.

$$DollarBond \quad B_t = e^{rt}$$

$$SterlingBond \quad D_t = e^{ut}$$

$$ExchangeRate \qquad C_t = C_0 \exp(\sigma W_t + \mu t)$$

here, W_t is Brownian motion, and r , u, σ and μ are constants. Although the first instrument is tradable, the exchange rate is not. The stochastic process C_t is the dollar value of a pound and clearly we can't trade with a foreign currency in our market. D_t is the price of a tradable, but is priced in pounds and therefore cannot be traded on our market. However, the product of the two, $S_T = C_t D_t$ is tradable on our market. An investor in America can hold sterling cash bonds with the dollar value by multiplying by the exchange rate C_t.

We can now follow a similar process to what we did for the basic Black-Scholes model. We will need to find a measure $\mathbb{Q}$ such that we can make

$$Z_t = C_0 \exp(\sigma W_t + (\mu + u - r)t)$$

into a martingale under the new measure. By the Cameron-Martin-Girsanov theorem, we will take

$$\tilde{W}_t = W_t + \sigma^{-1}\left(\mu + u - r + \frac{1}{2}\sigma^2\right)t$$

which allows the substitution to be made and we now have

$$Z_t = C_0 \exp\left(\sigma_t - \frac{1}{2}\sigma^2 t\right)$$

$$C_t = C_0 \exp\left(\sigma_t + (r - u - \frac{1}{2}\sigma^2)t\right)$$

With this, we will look at an example of a call option of buying a pound at time T for a future price of k dollars. Similarly to European call options, the payoff at time T is

$$X = (C_T - k)^+$$

where C_T is log-normally distributed. We can now introduce the Log-normal call formula with $\tilde{\sigma} = \sigma\sqrt{T}$:

$$E\left((F\,\exp(\tilde{\sigma}Z - \frac{1}{2}\tilde{\sigma}^2) - k\right) = F\Phi\Big(\frac{\log\frac{F}{k} + \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}}\Big) - k\Phi\Big(\frac{\log\frac{F}{k} - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}}\Big)$$

This is extremely similar to the Black-Scholes formula for European call options, but the F adds a little difficulty as we must compute $\mathbb{E}_{\mathbb{Q}}(C_T)$. Now, with our slight familiarity with how we handle derivatives based on foreign currency, lets look into quantos.

4.2 Quantos

A quanto is a derivative in which the underlying is based in a different currency than the contract is written in. Quanto options have both the strike price and underlying asset denominated in the foreign currency. If it is exercised, the value of the option is calculated in the foreign currency, which is then converted to our currency at the fixed exchange rate. To price quantos, we will look to a simple two-factor model. This model has a drift μ and v, positive volatilities σ_1 and σ_2, a correlation ρ lying between -1 and 1, and $\tilde{\rho} = \sqrt{1-\rho^2}$.

Quanto model

$$S_t = S_0 \exp(\sigma W_1(t) + \mu t), \qquad C_t = C_0 \exp(\rho\sigma_2 W_1(t) + \tilde{\rho}\sigma_2 W_2(t) + vt)$$

Like we saw before, there are three tradables: the dollar worth of the sterling bond C_tD_t, the dollar worth of the stock C_tS_t and the dollar cash bond B_t
With discounting the cash bond from the other tradables, we have $Y_t = B_t^{-1}C_tD_t$ and

$Z_t = B_t^{-1} C_t S_t$ and their SDEs are:

$$\begin{aligned} dY_t &= Y_t(\rho\sigma_2 dW_1(t) + \tilde{\rho}\sigma_2 dW_2(t) + (v + \frac{1}{2}\sigma_2^2 + u - r)dt) \\ dZ_t &= Z_t\big((\sigma_1 + \rho\sigma_2)dW_1(t) + \tilde{\rho}\sigma_2 dW_2(t) + (\mu + v + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2 - r)dt\big) \end{aligned}$$

As always, we need to find a measure to make them martingales. Since they are correlated, we must choose a vector such that both drift terms disappear at the same time. This vector is found by inverting the following matrix

$$\begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} = \begin{pmatrix} \rho\sigma_2 & \tilde{\rho}\sigma_2 \\ \sigma_1 + \rho\sigma_2 & \tilde{\rho}\sigma_2 \end{pmatrix} \begin{pmatrix} v + \frac{1}{2}\sigma_2^2 + u - r \\ \mu + v\frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2 - r \end{pmatrix}$$

and from this, we see that

$$\gamma_1 = \frac{\mu + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 - u}{\sigma_1} \qquad \gamma_2 = \frac{v + \frac{1}{2}\sigma_2^2 + u - r - \rho\sigma_2\gamma_1}{\tilde{\rho}\sigma_2}$$

Finally, we can rewrite the quanto model as

$$\begin{aligned} S_t &= S_0 \exp\big(\sigma_1 \tilde{W}_1(t) + (u - \rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^2)t\big) \\ C_t &= C_0 \exp\big(\rho\sigma_2 \tilde{W}_1(t) + \tilde{\rho}\sigma_2 \tilde{W}_2(t) + (r - u - \frac{1}{2}\sigma_2^2)t\big) \end{aligned}$$

Now that the dollar tradables are martingales, we can price up our quanto options. As we noted before, we need to price the forward contract in order to price the call. Our first step is to express the stock price at date T as:

$$S_T = \exp(-\rho\sigma_1\sigma_2 T) F \ \exp\left(\sigma_1\sqrt{T}Z - \frac{1}{2}\sigma_1^2 T\right)$$

with $F = S_0 e^{uT}$ and Z is a normally distributed random variable under $\mathbb{Q}$

At T = 0, the forward is equal to:

$$V_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T - k) = e^{-rT}\big(\exp(-\rho\sigma_1\sigma_2 T)F - k\big)$$

We have a requirement that this value must be equal to zero, since we cannot have arbitrage. So we set $k = F\ \exp(-\rho\sigma_1\sigma_2 T)$. To calculate this value, we write $F_Q = F\ \exp(-\rho\sigma_1\sigma_2 T)$ and

$$V_0 = e^{-rT}\Phi\left(\frac{\log\frac{F_Q}{k} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}\right)$$

and now we are ready to calculate the option price of $e^{-rT}\mathbb{E}_{\mathbb{Q}}((S_T - k)^+)$ as

$$V_0 = e^{-rT}\left(F_Q\Phi\Big(\frac{\log\frac{F_Q}{k} + \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}\Big) - k\Phi\Big(\frac{\log\frac{F_Q}{k} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}\Big)\right)$$

Chapter 5

Monte-Carlo Simulations for Foreign Options

We will follow a similar process as we did before, now for our foreign models.

5.1 Pricing Foreign Options in Matlab

Input needed: Stock price, Strike price, foreign interest rate, domestic interest rate, sigma 1 and sigma 2, end time, vector spacings, and the number of random walks you would like to compute - Stock, Strike, u, r, sig1, sig2, e, k, p respectively.

```
function [t,f,g,h, Wt]=QuantoIto(Stock,Strike,u,r,sig1,sig2,e,k,p)
t = linspace(0,e,k);
Wt = zeros(p,k);
rho = rand(1,1);
f = zeros(p,k);
g = zeros(p,1); % to take end values of f
h = zeros(1,p); % to take the column vector g and change it to a row vector
```

```
f(:,1) = [Stock];
for i = 2 : k
Wt(:,i) = Wt(:,i-1) + randn(p,1) * sdt;

f(:,i) = exp(-rho * sig1*sig2*t(i))*(Stock*exp(u*t(i))
                  *exp(sig1*sqrt(t(i))*Wt(:,i)-((.5)*sig1^(2)*t(i)));
end
g(:,1) = max(f(:,k)-Strike,0); % take values greater than end price
h(1,:) = g(:,1);               % use mean(h) to compute option value.
end
```

Again, we begin by plotting one random process. The result can be seen in Fig. 5.1. To evaluate the quanto via a Monte-Carlo simulation, we need to create an ensamble

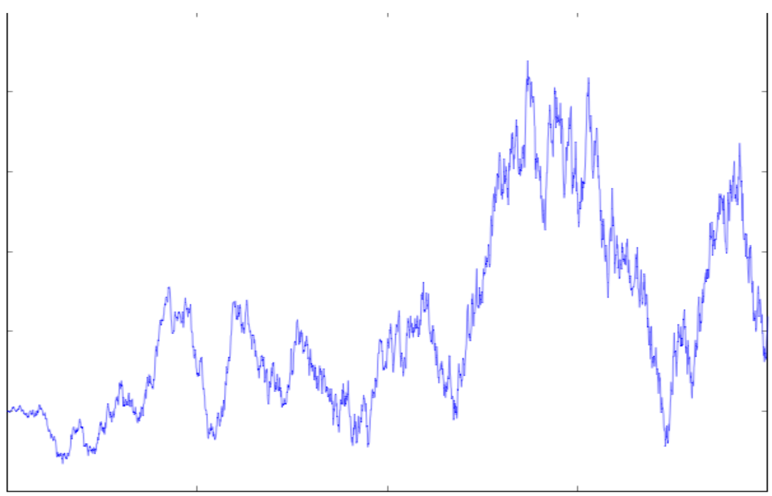

Figure 5.1: Single path for the quanto model.

of sample paths. Fig. 5.2 shows the plot 5000 paths. We expect the end values to follow a normal distribution. For this simulation, we used parameters that lead to smaller volatility within each process so we expect the walks to stay a little closer to center. Once again, at first look, everything seems to be modeled the way we want it

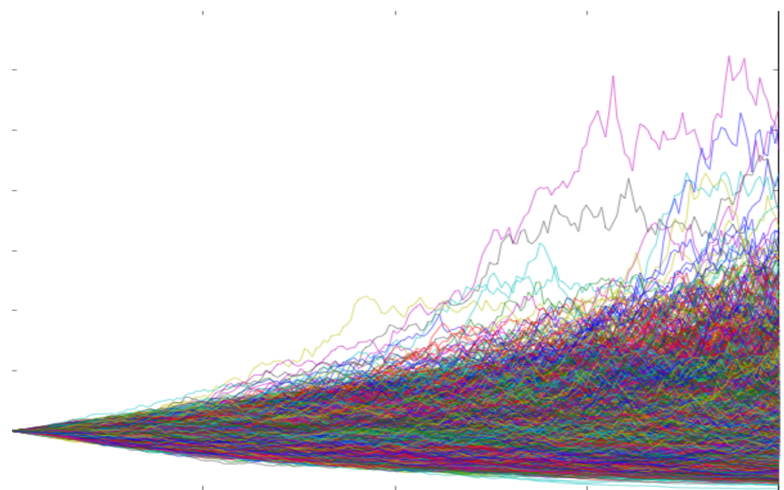

Figure 5.2: An ensemble of 5000 sample paths for the quanto model

to be. We will need to verify this with a histogram.

5.2 Checking the Distribution with a Bar Graph

We will be comparing our results on a histogram to check our distribution. We cannot expect it to look perfect as the bucket sizes will affect it. Yet again, We will be using Matlab's hist() command to produce our graph, except we will also be passing a parameter to signify 50 bins. As we can see from the figure below, the values closest to our mean were the most likely to occur, with the likely-hood decreasing as the deviation increases.Since our function has an exponential term in it, it will not drop below zero (much like the price of a stock), and therefore the values trailing off to the right is not a fundamental issue with our model.

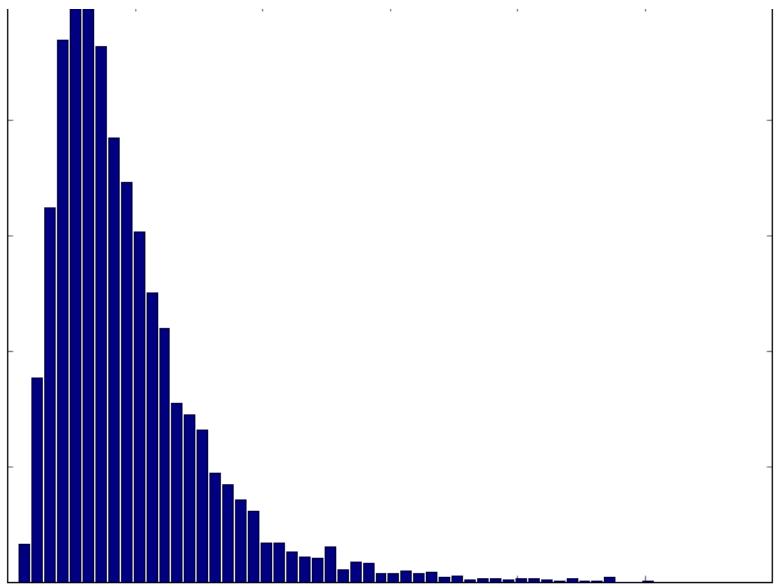

Figure 5.3: Bar graph showing the distribution of the end values for the quanto model

5.3 Checking our results with Black-Scholes

To calculate the price of an option, we first will write a small function to calculate F_Q We need

- u = foreign exchange interest rate
- T = time until expiration
- sig1,sig2 = positive volatilities
- Stock = initial stock price

```
function Fq = sterlingForward(Stock, u , T, sig1, sig2)
F = Stock * exp(u*T)  % basic sterling forward price
if rand(1,1) >.5      % rho is uniformly distributed between (-1,1)
   rho = rand(1,1)
else
   rho = -rand(1,1)
end
Fq = F *exp(-rho*sig1*sig2*T) % formula for the forward quanto
end
```

Now that we have F_Q we can use this function to find the price of the option:

```
function optionPrice = quantoOption(Stock, Strike, u, r, T, sig1,sig2)
Fq = sterlingForward(u,r,T,sig1,sig2)
d1 = (log(Fq / Strike)+ (.5)*(sig1^(2)) * T) / (sig1*sqrt(T))
d2 = (log(Fq / Strike)- (.5)*(sig1^(2)) * T) / (sig1*sqrt(T))

optionPrice = exp(-r*T) *((Fq)*normcdf(d1) - Strike * normcdf(d2))
```

We will use the parameters Stock = 100, Strike = 102, foreign interest rate = 1%, American Interest Rate = 5%, T = 2 years, sig1 = 20%, sig2 = 20%.

The result from the average of our Monte-Carlo simulation was $14.908 and using these parameters with our explicit formula we obtain $14.973 confirming the accuracy of our model – of course with the difference coming from our normally distributed random numbers.

5.4 Our Quanto Model and the Real World

Yet again, because of the level of simplicity with our model, we can not expect to be able to model actual foreign options at this point. In order to make the next step, we would need to investigate how to extract the correlation coefficient, handle when the stocks pay dividends, and then take all the other financial derivatives into account. In our model, we chose a simple random correlation coefficient to test our pricing algorithm based on a Monte-Carlo simulation. When we apply our method to the real world, this correlation coefficient should be obtained from a time series analysis of the underlying data (the exchange rate and the stock).

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